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Berry's phase in the Floquet representation

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Abstract. By means of the Floquet theory the investigation of Berry's phase for the quantum system with time-dependent periodic Hamiltonian is transformed into the study of a time-independent eigenvalue equation. An expression for the infinite-dimensional matrix is obtained describing the Berry's phase in terms of the Floquet state.

The topological phase for systems whose Hamiltonians $\mathcal{H}(t)$ depend periodically on time and whose evolution has a cyclic adiabatic character was introduced by Berry [1]. Most subsequent generalizations of this phase have been confined to such systems with a time-dependent Hamiltonian and have the prefix 'non-': non-Abelian [2], non-adiabatic [3], and non-cyclic [4].

Recently the Floquet theory has been used to study Berry's phase [5, 6]. However, it was only used to calculate the evolution matrix [6] or the periodic part of the evolution matrix [5] and the expression of topological connection in the Floquet representation has not yet been obtained.

In the present paper the Floquet theory is applied to the system whose Hamiltonian is periodic in time and whose evolution is described by a differential equation with time-dependent coefficients, and the investigation of Berry's phase for such a system is transformed into the study of a time-independent eigenvalue equation.

Let us consider a quantum system with a time-varying Hamiltonian whose dependence could be written in the form

$$\mathcal{H}(t) = V(t)\mathcal{H}_0 V^+(t) \tag{1}$$

with

$$V(t)V^+(t) = 1 \tag{2}$$

and the unitary operator $V(t)$ depends on the time cyclic, $V(t + t_c) = V(t)$, where t_c is the period.

The time-evolution operator $U(t)$ of the system satisfies the equation ($\hbar = 1$)

$$\mathrm{i}\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \mathcal{H}(t)U(t) \tag{3}$$

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with initial condition

$$U(0) = 1. \tag{4}$$

In order to find the Berry phase, let us subject the evolution operator to a unitary transformation [7]

$$\tilde{U}(t) = V^{+}(t)U(t) \tag{5}$$

then the equation (3) for the transformed operators has the form

$$\mathrm{i}\frac{\mathrm{d}\tilde{U}(t)}{\mathrm{d}t} = \left\{\mathrm{i}\frac{\mathrm{d}V^{+}(t)}{\mathrm{d}t}V(t) + \mathcal{H}_0\right\}\tilde{U}(t). \tag{6}$$

The extra term $\mathrm{i}\frac{\mathrm{d}V^{+}(t)}{\mathrm{d}t}V(t)$ reflects the topological properties and a simple calculation yields the expression for the Berry's phase [7]

$$\psi = -\mathrm{i}\int_0^{t_c}\left\langle \eta\left|\frac{\mathrm{d}V^{+}(t)}{\mathrm{d}t}V(t)\right|\eta\right\rangle \tag{7}$$

where $|\eta\rangle$ is an eigenvector of Hamiltonian $\mathcal{H}_0$:

$$\mathcal{H}_0|\eta\rangle = E_\eta|\eta\rangle. \tag{8}$$

On the other hand, the general form of the solution of a differential equation with periodic coefficients is given by Floquet theorem [8] and widely used in various fields of physics [8–12].

In the present problem, the periodicity of the Hamiltonian $\mathcal{H}(t)$ allows us to write the time-evolution operator $U(t)$ in Floquet form [8] as

$$U(t) = P(t)\mathrm{e}^{-\mathrm{i}tH} \tag{9}$$

where the unitary operator $P(t)$ has the same periodicity as $\mathcal{H}(t)$ and satisfies the equation

$$\mathrm{i}\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \mathcal{H}(t)P(t) - P(t)H \tag{10}$$

with initial condition

$$P(0) = 1 \tag{11}$$

and the time-independent operator H is defined by the expression

$$H = -\frac{\mathrm{i}}{t_c}\ln U(t_c). \tag{12}$$

But in order to find the time-independent equation we will not use the unitary transformation [13]. The next step is to expand the time-period operators $\mathcal{H}(t)$ and $P(t)$ in Fourier series [8, 9]

$$\mathcal{H}(t) = \sum_{n=-\infty}^{\infty}\mathcal{H}_n\mathrm{e}^{\mathrm{i}\omega_n t} \tag{13}$$

and

$$P(t) = \sum_{n=-\infty}^{\infty} P_n e^{i\omega_n t} \tag{14}$$

where $\omega_n = 2\pi n/t_c$. Substituting of these series into the time-dependent equation (10), we obtain the time-independent infinite-dimensional eigenvalue equation [8, 9]

$$\sum_k (\mathcal{H}_{n-k} + \omega_n \delta_{nk}) P_k = P_n H \tag{15}$$

with the infinite-dimensional time-independent Floquet Hamiltonian [8, 9]

$$\mathcal{H}^{\mathrm{F}}_{n-k} = \mathcal{H}_{n-k} + \omega_n \delta_{nk}. \tag{16}$$

Now, in order to separate the topological part from the time-independent equation (15), let us use the unitary transformation. It begins by expanding the unitary operators $V(t)$ and $V^+(t)$ in Fourier series:

$$V(t) = \sum_{n=-\infty}^{\infty} = V_n e^{i\omega_n t} \tag{17}$$

and

$$V^+(t) = \sum_{n=-\infty}^{\infty} V_{-n} e^{i\omega_n t}. \tag{18}$$

It is easy to show that the Fourier components V_n and V_n^+ satisfy the following expression:

$$\sum_k V^+_{n-k} V_{k-m} = \delta_{nm}. \tag{19}$$

Transforming (15) with the aid of the Fourier components V_n, i.e. substituting

$$P_n = \sum_{k=-\infty}^{\infty} V_{k-n} \tilde{P}_k \tag{20}$$

gives the transformed equation

$$\sum_{k=-\infty}^{\infty} \tilde{\mathcal{H}}^{\mathrm{F}}_{n-k} \tilde{P}_k = \tilde{P}_n H \tag{21}$$

where

$$\tilde{\mathcal{H}}^{\mathrm{F}}_{n-k} = \Phi_{n-k} + \delta_{nm}(\omega_n + \mathcal{H}_0) \tag{22}$$

and the infinite-dimensional matrix Φ_{n-k} reproduces Berry's connection in the Floquet representation

$$\Phi_{n-k} = \sum_{m=-\infty}^{\infty} \omega_{m-n} V^+_{n-m} V_{m-k}. \tag{23}$$

Equation (23) is the main result of this paper.

The time-evolution operator $\tilde{U}(t)$ can be expressed in the Floquet states [8] by

$$\tilde{U}_{\alpha\beta}(t) = \sum_{n=-\infty}^{\infty} \langle \alpha n | e^{-i\tilde{\mathcal{H}}t} | 0\beta \rangle e^{i\omega_n t} \tag{24}$$

where indexes α and β represent an atomic state [8]. According to the adiabatic limit, it is correct to retain only the diagonal part of the Hamiltonian (22) and an expression for Berry's phase in the Floquet representation is of the form

$$\psi = t_c \sum_{m=-\infty}^{\infty} \langle \alpha | V_{n-m}^{+} V_{m-n} | \alpha \rangle. \tag{25}$$

As an example, let us consider a particle with spin $\boldsymbol{S}$ in an adiabatically precessing magnetic field $\boldsymbol{H}(t)$, having constant magnitude. The Hamiltonian is

$$\mathcal{H}(t) = \gamma \boldsymbol{H}(t) \boldsymbol{S} \tag{26}$$

where

$$\boldsymbol{H}(t) = \boldsymbol{H}_0 \{ \boldsymbol{k} \cos\theta + \boldsymbol{i} \sin\theta \cos\phi(t) + \boldsymbol{j} \sin\theta \sin\phi(t) \}. \tag{27}$$

Here θ and $\phi(t) = \omega t$ indicate the direction of the magnetic field. Hamiltonian (26) can be written in the form

$$\mathcal{H}(t) = \omega_0 V(t) S_z V^{+}(t) \tag{28}$$

where $\omega_0 = \gamma |\boldsymbol{H}_0|$ and

$$V(t) = e^{i\omega t S_z} e^{i\theta S_y} e^{-i\omega t S_z} \tag{29}$$

is a unitary operator with time period $t_c = 2\pi/\omega$. If the adiabatic condition is fulfilled, $\omega \ll \omega_0$, then only the diagonal part should be taken into account and simple calculations yield, from equation (25), the Berry's phase

$$\psi = \pi(1 - \cos\theta). \tag{30}$$

This result was obtained early [1].

We close by noting that the topological part of the general phase can be found for the system described by the time-independent equation. It would also be useful to extend the unitary transformation (20) to a related problem, for example to find the interaction representation [12] after Fourier transformation.

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